On a system of nonlinear ordinary differential equations containing a small parameter

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§ 1. Introduction.

In this note we consider a differential equation of the form

$$\Lambda(\varepsilon)\frac{d\vec{y}}{dx}=\vec{a}(x,\varepsilon)+A(x,\varepsilon)\vec{y}+\vec{f}(x,\varepsilon,y)\ ,\tag{1.1}$$

where we suppose that

1) $\Lambda(\varepsilon)$ is an n-by-n diagonal matrix whose elements are $\varepsilon^{\sigma_1},\varepsilon^{\sigma_2},\cdots,\varepsilon^{\sigma_n}$:

$$\Lambda(\varepsilon)=\begin{pmatrix}\varepsilon^{\sigma_1} & & & \\ & \varepsilon^{\sigma_2} & & 0 \\ & 0 & \ddots & \\ & & & \varepsilon^{\sigma_n}\end{pmatrix}$$

with nonnegative integers σ_i;

2) x and ε are a complex independent variable and a complex parameter respectively;

3) $\vec{y}$ denotes an n-dimensional column vector.

4) $\vec{a}(x,\varepsilon)$ is an n-dimensional column vector whose components are functions holomorphic and bounded in (x,ε) for

$$|x|<a,\ 0<|\varepsilon|<b,\ |\arg\varepsilon|<\theta\tag{1.2}$$

5) $A(x,\varepsilon)$ is an n-by-n matrix those components are functions holomorphic and bounded in (x,ε) for (1.2)

6) $\vec{f}(x,\varepsilon,y)$ is an n-dimensional column vector of the form

$$\vec{f}(x,\varepsilon,y)=\sum{}'' f_{\mathfrak{L}}(x,\varepsilon)y^{\mathfrak{L}}\ ,\tag{1.3}$$

$$\mathfrak{L}=(l_1,l_2,\cdots,l_n)\ ,\tag{1.4}$$

$$y^{\mathfrak{L}}=y_1^{l_1}y_2^{l_2}\cdots y_n^{l}\ ,\tag{1.5}$$

where the power series in the right-hand member is uniformly convergent for

$$|x|<a,\ 0<|\varepsilon|<b,\ |\arg\varepsilon|<\theta,\ \|\vec{y}\|<c,^{1)}\tag{1.6}$$

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1) $\|\vec{y}\|=\max\{|y_1|,\cdots,|y_n|\}$

and $\sum''$ denotes the summation taken all over the arrangements $\mathfrak{L}$ of n non-negative integers l_i satisfying the inequality

$$(1.7) \qquad |\mathfrak{L}|=\sum_{i=1}^{n} l_i \geqq 2\ ;$$

7) $\vec{a}(x, \varepsilon)$, $A(x, \varepsilon)$ and $\vec{f}_{\mathfrak{L}}(x, \varepsilon)$ admit uniformly asymptotic expansions in powers of ε for

$$(1.8) \qquad |x|<a\ ,$$

$$(1.9) \qquad \vec{a}(x, \varepsilon) \sim \sum_{k=0}^{\infty} \varepsilon^k \vec{a}_k(x)\ ,$$

$$(1.10) \qquad A(x, \varepsilon) \sim \sum_{k=0}^{\infty} \varepsilon^k A_k(x)\ ,$$

$$(1.11) \qquad f_{\mathfrak{L}}(x, \varepsilon) \sim \sum_{k=0}^{\infty} \varepsilon^k \vec{f}_{\mathfrak{L},k}(x)\ ,$$

as ε tends to 0 in the domain

$$(1.12) \qquad 0<|\varepsilon|<b,\ |\arg \varepsilon|<\theta\ .$$

The components of n-dimensional vectors $\vec{a}_k(x)$, $\vec{f}_{\mathfrak{L},k}(x)$ and n-by-n matrices $A_k(x)$ are functions holomorphic and bounded for (1.8);

8) a matrix $\tilde{A}_0(x)$ formed by the elements corresponding to the jth row and lth column of the matrix $A_0(x)$ such that $\sigma_j>0, \sigma_l>0$, is non-singular for (1.8) and is triangular form for $x=0$.

The aim of this note is to construct an expression of solutions bounded and containing several arbitrary constants.

I. Preliminaries

Recently, M. Iwano has studied the same problem in [3] and he obtained three systems of particular solutions which contain several arbitrary constants and converge to zero as ε tends to 0 in an angular region contained in (1.12). Our method is essentially similar to his method. We explain the results obtained in [3].

§ 2. Reduction of the system (1.1), I.

In case when the value of $\vec{a}_0(0)$ is equal to $\vec{0}$ or the right-hand member of (1.1) is a polynomial of $y_1, y_2, \cdots, y_n$, we may suppose without loss of generality that:

$$(2.1) \qquad \vec{a}(x, \varepsilon) \equiv \vec{0}\ .$$

This assertion is based on the following

Theorem A. (Theorem 1 in [3]). *There exists a solution*

$$\vec{y}=\vec{p}(x, \varepsilon) , \tag{2.2}$$

where $\vec{p}(x, \varepsilon)$ *is an* n*-dimensional column vector whose components are functions holomorphic and bounded in* (x, ε) *for*

$$|x|<a',\ 0<|\varepsilon|<b',\ |\arg \varepsilon|<\theta' \tag{2.3}$$

and admitting a uniformly asymptotic expansion in powers of ε:

$$\vec{p}(x, \varepsilon)\sim \sum_{k=0}^{\infty} \varepsilon^k \vec{p}_k(x) \tag{2.4}$$

as ε *tends to* 0 *in the domain* (2.3), *where* $\vec{p}_k(x)$ *are* n*-dimensional column vector functions holonorphic and bounded for* $|x|<a'$. *Here* a', b' *and* θ' *are suitably chosen positive constants.*

Owing to this theorem, we see immediately that the transformation

$$\vec{y}=\vec{p}(x, \varepsilon)+\vec{z}$$

yields a system without the term which is independent of $\vec{z}$.

§ 3. Reduction of the system (1.1), II.

Let $a_{jh}(x, \varepsilon)$ be the element on the j^{th} row and the hth column of the matrix $A(x, \varepsilon)$.

Then, in our following discussion, no generality is lost in assuming that "*if for the index* (j, h) *one of the following conditions is satisfied*:

$$(\sigma_j\neq\sigma_h) \quad \textit{or} \quad (\sigma_j=\sigma_h \quad \textit{and} \quad a_{0\cdot jj}(0)\neq a_{0\cdot hh}(0))$$

or $(\sigma_j=\sigma_h=0)$, *then*

$$a_{jh}(x, \varepsilon)\equiv 0 \text{ .''} \tag{3.1}$$

This assertion is based on the following:

Let $P(x, \varepsilon)$ be an n-by-n matrix whise components are functions holomorphic and bounded for

$$|x|<a'',\ 0<|\varepsilon|<b'',\ |\arg \varepsilon|<\theta'' \tag{3.2}$$

and admitting a uniformly asymptotic expansion in powers of ε:

$$P(x, \varepsilon)\sim \sum_{k=0}^{\infty} \varepsilon^k P_k(x) \tag{3.3}$$

as ε tends to 0 in the domain (3.2), where $P_k(x)$ are n-by-n matrix functions holomorphic and bounded for $|x|<a''$.

Theorem B (Theorem 2 in [3]). *If we choose the positive constants a'', b'' and θ'' in a suitable way, there exists a matrix function $P(x, \varepsilon)$ satisfying the conditions given above such that the system* (1.1) *is reduced to*

$$\Delta(\varepsilon)\frac{d\vec{z}}{dx}=B(x, \varepsilon)\vec{z}+\vec{g}(x, \varepsilon, \vec{z}) \tag{3.4}$$

by a transformation

$$\vec{y}=P(x, \varepsilon)\vec{z}\ ; \tag{3.5}$$

where

i) $B(x, \varepsilon)$ *in an n-by-n matrix function with the same properties as the matrix* $A(x, \varepsilon)$;

ii) *the* (j, h)*-element* $b_{jh}(x, \varepsilon)$ *of* $B(x, \varepsilon)$ *satisfies the conditions*

$$b_{jh}(x, \varepsilon)\equiv 0 \tag{3.6}$$

if $(\sigma_j\neq\sigma_h)$ *or* $(\sigma_j=\sigma_h$ *and* $b_{0,jj}(0)\neq b_{0,hh}(0))$ *or* $(\sigma_j=\sigma_h=0)$;

iii) *the matrix formed by* $b_{0,jh}(x)(\sigma_j>0, \sigma_h>0)$ *is non-singular for* $|x|<a''$ *and is triangular for* $x=0$. *But the values* $b_{0,jj}(0)$ *do not coincide, in general, with* $a_{0,jj}(0)$.

iv) $\vec{g}(x, \varepsilon, z)$ *is an n-dimensional vector function with same properties as the vector function* $\vec{f}(x, \varepsilon, y)$.

§ 4. Solutions depending on several arbitrary constants.

Owing to Theorems A and B, we can suppose that the system (1.1) satisfies the conditions (2.1) and (3.1).

Let

$$\sigma_j\begin{cases}=0 & (j=n', n'+1, \cdots, n'')\ ,\\ >0 & (j<n' \ \text{ or } \ j>n'')\end{cases} \tag{4.1}$$

and denote by $\lambda_j\,(j=1, 2, \cdots, n)$ the diagonal elements of the matrix $\tilde{A}_0(0)$.

Since the quantities $\lambda_j\,(\sigma_j>0)$ are supposed to be different from zero, we can choose a real number Ω so that inequalities

$$Re\frac{\lambda_1 e^{-i\Omega}}{\varepsilon^{\sigma_1}}\geqq\cdots\geqq Re\frac{\lambda_{n'-1}e^{-i\Omega}}{\varepsilon^{\sigma_{n'-1}}}=\cdots>0>Re\frac{\lambda_{n''+1}e^{-i\Omega}}{\varepsilon^{\sigma_{n''+1}}}\geqq\cdots\geqq Re\frac{\lambda_n e^{-i\Omega}}{\varepsilon^{\sigma_n}} \tag{4.2}$$

hold for

$$0<|\varepsilon|<b''\ , \qquad \arg\varepsilon=0$$

and

(4.3) $$\sigma_1=\cdots=\sigma_\alpha>\sigma_{\alpha+1}\geqq\cdots\geqq\sigma_{n'-1}>0\ ,\qquad 0<\sigma_{n''+1}\leqq\cdots\leqq\sigma_{\beta-1}<\sigma_\beta=\cdots=\sigma_n\ .$$

Clearly these two inequalities are consistent.

In our following discussion, without loss of generality, we assume that these two inequalities are satisfied.

Let

(4.4) $$\mu_j=\begin{cases} Re[\lambda_j e^{-i\Omega} b_0^{-\sigma_j}] & (j<n')\ ,\\ 1 & (n'\leqq j\leqq n'')\ ,\\ -Re[\lambda_j e^{-i\Omega} b_0^{-\sigma_j}] & (n''<j)\ .\end{cases}$$

We denote by $(\gamma'_\kappa, \cdots, \gamma''_\kappa)$

$$(1, \cdots, \alpha) \quad \text{or} \quad (n', \cdots, n'') \quad \text{or} \quad (\beta, \cdots, n)$$

according as $\kappa=1$ or $\kappa=2$ or $\kappa=3$, and, to simplify the description, dropping the subscript κ.

Then, the main result obtained in [3] is written as follows:

Theorem C (Theorem 4 in [3]). *The system* (1.1) *admits three sets of solutions of the form*

$$(F^{(\kappa)})\begin{cases} \vec{y}=\vec{\phi}^{(\kappa)}(x, \varepsilon, U_{\gamma'}, \cdots, U_{\gamma''}) & (\kappa=1, 2, 3)\ ,\\ U_j\equiv U_j(x, \varepsilon; x_0, u^0_{\gamma'}, \cdots, u^0_{\gamma''}) & (j=\gamma', \cdots, \gamma'')\ ,\end{cases}$$

where

i) $\vec{\phi}^{(\kappa)}(x, \varepsilon, u)$ *is an n-dimensional vector function holomorphic and bounded in* (x, ε, u) *for*

(4.5) $$|x|<a_0,\ 0<|\varepsilon|<b_0,\ |\arg\varepsilon|<\theta_0,\ \max_{k=\gamma'}^{\gamma''}|u_k|^{1/\mu_k}<c_0$$

and developable in a uniformly convergent power series of $u_\gamma, \cdots, u_{\gamma''}$:

(4.6) $$\vec{\phi}^{(\kappa)}(x, \varepsilon, u)=\sum_{|\mathfrak{K}|\geqq 1}\vec{\phi}^{(\kappa)}_{\mathfrak{K}}(x, \varepsilon)u^{\mathfrak{K}}$$
$$\mathfrak{K}=(k_{\gamma'}, \cdots, k_{\gamma''})$$
$$u^{\mathfrak{K}}=u_{\gamma'}^{\kappa_{\gamma'}}\cdots u_{\gamma''}^{\kappa_{\gamma''}}$$

where $\vec{\phi}^{(\kappa)}_{\mathfrak{K}}(x, \varepsilon)$ *are n-dimensional vector functions holomorphic and bounded in* (x, ε) *for*

(4.7) $$|x|<a_0,\ 0<|\varepsilon|<b_0,\ |\arg\varepsilon|<\theta_0$$

and admitting uniformly asymptotic expansions in powers of ε *as* ε *tends to* 0 *in the domain* (4.7) (*whose coefficients are n-dimensional vector functions holomorphic and bounded for* $|x|<a_0$);

ii) U_γ $(\gamma=\gamma', \cdots, \gamma'')$ *is a solution of a reduced form:*

$$\varepsilon^{\sigma_\gamma}\frac{du_\gamma}{dx}=\sum a_{\gamma h}(x, \varepsilon)u_h+\sum{}' g_{\gamma,\kappa_{\gamma'}\cdots\kappa_{\gamma''}}(x, \varepsilon)u_{\gamma'}^{k_{\gamma'}}\cdots u_{\gamma''}^{\kappa_{\gamma''}} \quad (\gamma=\gamma', \cdots, \gamma'') \tag{4.8}$$

or

$$\frac{du_\gamma}{dx}=0 \qquad (\gamma=\gamma', \cdots, \gamma'') \tag{4.8'}$$

according as $(\gamma_{,}, \cdots, \gamma'')\neq(n', \cdots, n'')$ *or* $(\gamma', \cdots, \gamma'')=(n', \cdots, n'')$, *satisfying an initial condition*

$$u_j^0=U_j(x_0, \varepsilon; x_0, u_{\gamma'}^0, \cdots, u_{\gamma''}^0)\ , \tag{4.9}$$

where the summation $\sum'$ *should be taken all over the arrangements* $(l_{\gamma'}, \cdots, l_{\gamma''})$ *such that*[2)]

$$l_{\gamma'}+\cdots+l_{\gamma''}\geqq 2 \quad \textit{and} \quad l_{\gamma'}\lambda_{\gamma'}+\cdots+l_{\gamma''}\lambda_{\gamma''}=\lambda_\gamma\ .$$

§ 5. Extension of Theorem C.

Our aim of this paper is to extend Theorem C in a following way:
We construct two sets of particular solution

$$\vec{y}=\vec{\Psi}_+(x, \varepsilon; \tilde{U}_1, \cdots, \tilde{U}_\alpha, \tilde{U}_{n'}, \cdots, \tilde{U}_{n''}) \tag{F_+}$$

and

$$\vec{y}=\vec{\Psi}_-(x, \varepsilon; \tilde{U}_{n'}, \cdots, \tilde{U}_{n''}, \tilde{U}_\beta, \cdots, \tilde{U}_n) \tag{F_-}$$

where

i) $\tilde{U}_j$ is a holomorphic and bounded solution of a certain reduced system satisfying the same initial condition as U_j;

ii) $\vec{\Psi}_+$ and $\vec{\Psi}_-$ are functions developable in uniformly convergent power series of $\tilde{U}_1, \cdots, \tilde{U}_\alpha, \tilde{U}_{n'}, \cdots, \tilde{U}_{n''}$ and $\tilde{U}_{n'}, \cdots, \tilde{U}_{n''}, \tilde{U}_\beta, \cdots, \tilde{U}_n$ respectively. If we put

$$u_j^0=0 \quad (j=1, \cdots, \alpha \quad \text{or} \quad \beta, \cdots, n)\ ,$$

$\vec{\Psi}_+(x,\varepsilon, U)$ (or $\vec{\Psi}_-(x, \varepsilon, U)$) is reduced to the function $\vec{\phi}^{(2)}(x, \varepsilon, U_{n'}, \cdots, U_{n''})$, and if we put

$$u_j^0=0 \qquad (j=n', \cdots, n'')$$

the functions $\vec{\Psi}_+(x, \varepsilon, U)$ and $\vec{\Psi}_-(x, \varepsilon, U)$ are reduced to the functions $\vec{\phi}^{(1)}(x, \varepsilon, U)$ and $\vec{\phi}^{(3)}(x, \varepsilon, U)$ aespectively.

2) Owing to the inequalities (4.2) and (4.3) we see immediately that the right-hand members of the reduced form are polynomials of $u_{\gamma'}, \cdots, u_{\gamma''}$.

II Reduction of the nonlinear part

§ 7. Assumptions.

In this chapter we transform the system (1.1) into a form as simple as possible. We begin with the utterly formal construction of transformation. Its convergence will be shown in the latter chapters.

In virtue of Theorems A and B, instead of system (1.1), it is sufficient to consider the following system:

$$\varepsilon^{\sigma_j}\frac{dy_j}{dx}=\sum_l a_{jl}(x,\varepsilon)y_l+\sum_{|\mathfrak{K}|\geqq 2} a_{j\mathfrak{K}}(x,\varepsilon)y_1^{\kappa_1}\cdots y_n^{\kappa_n}\qquad (j=1,2,\cdots,n) \tag{7.1}$$

where we suppose that

i) $a_{jl}(x,\varepsilon)$ and $a_{j\mathfrak{K}}(x,\varepsilon)$ are functions holomorphic and bounded in (x,ε) for

$$|x|<a,\ 0<|\varepsilon|<b,\ |\arg\varepsilon|<\theta\ , \tag{7.2}$$

and admitting uniformly asymptotic expansions in powers of ε as ε tends to zero in the domain (7.2).

ii) $\lambda_j(\equiv a_{jj,0}(0))$ and σ_j satisfy the inequalities (4.2) and (4.3) respectively.

iii) if $(\sigma_j\neq\sigma_l)$ or $(\sigma_j=\sigma_l,\ \lambda_j\neq\lambda_l)$ or $(\sigma_j=\sigma_l=0)$, then we have

$$a_{jl}(x,\varepsilon)\equiv 0 \tag{7.3}$$

and the matrix formed by $a_{jl,0}(x)(\sigma_j>0,\ \sigma_l>0)$ is non-singular for $|x|<a$ and the matrix is of Jordan's canonical form for $x=0$.

iv) the power series in the right hand side of (7.1) is uniformly convergent for

$$|x|<a,\ 0<|\varepsilon|<b,\ |\arg\varepsilon|<\theta,\ \|\vec{y}\|<c\ .$$

§ 8. Formal transformation.

For simplicity's sake, hereafter we adopt the following notations:

$$\mathfrak{L}=(l_1,\cdots,l_n)\qquad (l_i\geqq 0 \text{ are integers})\ ,$$

$$(S_\alpha)\begin{cases}(\kappa',\cdots,\kappa'')=(1,\cdots,\alpha,n',\cdots,n'')\ ,\\(\gamma',\cdots,\gamma'')=(1,\cdots,\alpha)\ ,\\\quad\mathfrak{L}_\kappa=(l_{\kappa'},\cdots,l_{\kappa''})\equiv(l_1,\cdots,l_\alpha,l_{n'},\cdots,l_{n''})\\\quad\mathfrak{L}_\gamma=(l_{\gamma'},\cdots,l_{\gamma''})\equiv(l_1,\cdots,l_\alpha)\ ,\\\quad\mathfrak{L}'=(l_1,\cdots,l_\alpha,0,\cdots,0,l_{n'},\cdots,l_{n''},0,\cdots,0)\ ,\\\quad\mathfrak{L}'_\delta=(\delta_{1h},\cdots,\delta_{\alpha h},0,\cdots,0,l_{n'},\cdots,l_{n''},0,\cdots,0)\ ,\end{cases}$$

or

$$
(S_\beta)\begin{cases}
(\kappa', \cdots, \kappa'')=(n', \cdots, n'', \beta, \cdots, n) , \\
(\gamma', \cdots, \gamma'')=(\beta, \cdots, n) , \\
\quad \mathfrak{L}_\kappa=(l_{n'}, \cdots, l_{n''}, l_\beta, \cdots, l_n) , \\
\quad \mathfrak{L}_\gamma=(l_\beta, \cdots, l_n) \\
\quad \mathfrak{L}'=(0, \cdots, 0, l_{n'}, \cdots, l_{n''}, 0, \cdots, 0, l_\beta, \cdots, l_n) , \\
\quad \mathfrak{L}'_\delta=(0, \cdots, 0, l_{n'}, \cdots, l_{n''}, 0, \cdots, 0, \delta_{\beta h}, \cdots, \delta_{nh}) ,
\end{cases}
$$

where δ_{jh} is the Kronecker's symbol.

Let us consider a formal transformation

$$
(8.1) \qquad y_j \sim z_j + \sum_{|\mathfrak{L}_\kappa| \geqq 2} p_{j\mathfrak{L}_\kappa}(x, \varepsilon) z_{\kappa'}^{l_{\kappa'}} \cdots z_{\kappa''}^{l_{\kappa''}} , \qquad j=1, 2, \cdots, n .
$$

We impose upon coefficients $p_{j\mathfrak{L}_\kappa}(x, \varepsilon)$ following conditions: $p_{j\mathfrak{L}_\kappa}(x, \varepsilon)$ are functions holomorphic and bounded in (x, ε) for

$$
(8.2) \qquad |x|<a', \ 0<|\varepsilon|<b'', \ |\arg \varepsilon|<\theta' ,
$$

and admit uniformly asymptotic expansions

$$
p_{j\mathfrak{L}_\kappa}(x, \varepsilon) \sim \sum_{h=1}^{\infty} \varepsilon^h p_{j\mathfrak{L}_\kappa \cdot h}(x)
$$

as ε tends to zero in the domain (8.2).

Applying (8.1) to the system (7.1), it will be transformed formally into the system

$$
(8.3) \qquad \varepsilon^{\sigma_j} \frac{dz_j}{dx} \sim \sum_{l=1}^{n} a_{jl}(x, \varepsilon) z_l + \sum_{|\mathfrak{L}| \geqq 2} b_{j\mathfrak{L}}(x, \varepsilon) z_1^{l_1} \cdots z_n^{l_n} .
$$

Then, without any essential modification of the reasonings used in the proof of the Theorem 3 in [3], we can prove the following:

Theorem 1. *We can determine the coefficients $p_{j\mathfrak{L}_\kappa}(x, \varepsilon)$ satisfying the conditions given above, such that $b_{j\mathfrak{L}}(x, \varepsilon)$ is a functions holomorphic and bounded in (x, ε) for* (8.2) *and admitting a uniformly asymptotic expansion in powers of ε as ε tends to* 0 *in the domain* (8.2), *and, especially,*

$$
(8.4) \qquad b_{j\mathfrak{L}'}(x, \varepsilon) \equiv 0
$$

if $\{j \neq \gamma', \cdots, \gamma''\}$ or $\{j=\gamma', \cdots, \gamma''$ and $\lambda_j \neq l_{\kappa'}\lambda_{\kappa'} + \cdots + l_{\kappa''}\lambda_{\kappa''}\}$

Remark. In virtue of this Theorem, if the coefficients $b_{j\mathfrak{L}'}(x, \varepsilon)$ do not vanish identically, the following relations should hold:

$$
j=\gamma', \cdots, \gamma'' \quad \text{and} \quad \lambda_j = l_{\kappa'}\lambda_{\kappa'} + \cdots + l_{\kappa''}\lambda_{\kappa''} .
$$

Therefore, owing to the inequalities (4.2), the above relations yield:

$$\sigma_j=\sigma_1 \qquad \lambda_j=l_{j+1}\lambda_{j+1}+\cdots+l_\alpha\lambda_\alpha$$

or

$$\sigma_j=\sigma_n \qquad \lambda_j=l_\beta\lambda_\beta+\cdots+l_{j-1}\lambda_{j-1}\,.$$

III Reduced system and formal solutions.

§ 9. Formal reduced system and auxiliary formal solution.

For brevity we shall adopt a notation $(8.3)_j$ to signify the jth equation of the system (8.3).

From Theorem 1, immediately results the following:

Lemma 1. *If we put in system* (8.3)

$$z_k=0\,, \qquad k\neq\kappa', \cdots, \kappa''\,,$$

then the equation $(8.3)_j$ $(j\neq\kappa', \cdots, \kappa'')$ *is satisfied and the resting system will be formally reduced to a system*

$$(9.1)\qquad \begin{cases} \varepsilon^{\sigma_j}\dfrac{dz_j}{dx}=\sum\limits_{l=\gamma'}^{\gamma''} a_{jl}(x,\varepsilon)z_l+\sum b_{j\mathfrak{K}_\kappa}(x,\varepsilon)z_{\kappa'}^{k_{\kappa'}}\cdots z_{\kappa''}^{k_{\kappa''}}, & j=\gamma', \cdots, \gamma'' \\ \dfrac{dz_j}{dx}=0\,, & j=n', \cdots, n''\,. \end{cases}$$

In order that $b_{j\mathfrak{K}_\kappa}(x,\varepsilon)$ does not vanish identically, we must have

$$\lambda_j-k_{\kappa'}\lambda_{\kappa'}-\cdots-k_{\kappa''}\lambda_{\kappa''}=0$$

Since $\lambda_{n'}=\cdots=\lambda_{n''}=0$, there exist infinitely many such arrangements $(k_{\kappa'}, \cdots, k_{\kappa''})$. However the arrangements $(k_{\gamma'}, \cdots, k_{\gamma''})$ are obviously only finite in number. Therefore, we can arrange the formal power series on the right hand side of (9.1) in the form of polynomials of $z_{\gamma'}, \cdots, z_{\gamma''}$. Thus the formal reduced form (9.1) can be written as

$$(9.2)_1 \qquad \varepsilon^{\sigma_j}\frac{dz_j}{dx}=\sum_{h=\gamma'}^{\gamma''}\{a_{jh}(x,\varepsilon)+c_{jh}(x,\varepsilon,z_{n'},\cdots,z_{n''})\}z_h$$
$$+\sum_{L_\gamma} c_{j\mathfrak{L}_\gamma}(x,\varepsilon,z_{n'},\cdots,z_{n''})z_{\gamma'}^{l_{\gamma'}}\cdots z_{\gamma''}^{l_{\gamma''}} \qquad j=\gamma', \cdots, \gamma''\,,$$

$$(9.2)_2 \qquad \frac{dz_j}{dx}=0 \qquad\qquad j=n', \cdots, n''\,.$$

On the other hand, by putting $z_k=0$ $(k\neq\kappa', \cdots, \kappa'')$ in (8.1), we can arrange formally in the form

$$(9.3)\qquad y_j \sim \phi_j(x, \varepsilon, z_{n'}, \cdots, z_{n''}) + \sum_{h=\gamma'}^{\gamma''} \{\sigma_{jh} + \phi_{jh}(x, \varepsilon, z_{n'}, \cdots, z_{,,})\} z_h$$
$$+ \sum_{|\mathfrak{L}_\gamma| \geqq 2} \phi_{j\mathfrak{L}_\gamma}(x, \varepsilon, z_{n'}, \cdots, z_{n''}) z_{n'}^{l_{n'}} \cdots z_{n''}^{l_{n''}} \qquad (j=1, 2, \cdots, n)$$

where

$$(9.4)\qquad \phi_j \sim \delta_j^* z_j + \sum p_{j\mathfrak{L}_\delta}(x, \varepsilon) z_{n'}^{l_{n'}} \cdots z_{n''}^{l_{n''}}$$

$$\delta_j^* = \begin{cases} 1 & j=n', \cdots, n'' \\ 0 & j \neq n', \cdots, n''\ ; \end{cases}$$

$$(9.5)\qquad \phi_{jh} \sim \begin{cases} \sum p_{j\mathfrak{L}_\delta}(x, \varepsilon) z_{n'}^{l_{n'}} \cdots z_{n''}^{l_{n''}} & (\lambda_j \neq \lambda_h)\ , \\ 0 & (\lambda_j = \lambda_h)\ , \end{cases} \qquad \begin{pmatrix} h=\gamma', \cdots, \gamma'', \\ j=1, 2, \cdots, n, \end{pmatrix},$$

$$(9.6)\qquad \phi_{i\mathfrak{L}_\gamma} \sim \begin{cases} \sum p_{j\mathfrak{L}'}(x, \varepsilon) z_{n'}^{l_{n'}} \cdots z_{n''}^{l_{n''}} & (\sigma_j \neq \sigma_\gamma \text{ or } \lambda_j \neq l_{\gamma'}\lambda_{\gamma'} + \cdots + l_{\gamma''}\lambda_{\gamma''})\ , \\ 0 & (\sigma_j = \sigma_\gamma \text{ and } \lambda_j = l_{\gamma'}\lambda_{\gamma'} + \cdots + l_{\gamma''}\lambda_{\gamma''})\ , \end{cases}$$
$$j=1, 2, \cdots, n\ ;$$

$$(9.7)\qquad c_{jh} \sim \begin{cases} \sum b_{j\mathfrak{L}_\delta}(x, \varepsilon) z_{n'}^{l_{n'}} \cdots z_{n''}^{l_{n''}} & (\lambda_j = \lambda_h)\ , \\ 0 & (\lambda_j \neq \lambda_h)\ , \qquad j, h=\gamma', \cdots, \gamma''\ ; \end{cases}$$

$$(9.8)\qquad c_{j\mathfrak{L}_\gamma} \sim \begin{cases} \sum b_{j\mathfrak{L}'}(x, \varepsilon) z_{n'}^{l_{n'}} \cdots z_{n''}^{l_{n''}} & (\lambda_j = l_{\gamma'}\lambda_{\gamma'} + \cdots + l_{\gamma''}\lambda_{\gamma''})\ , \\ 0 & (\lambda_j \neq l_{\gamma'}\lambda_{\gamma'} + \cdots + l_{\gamma''}\lambda_{\gamma''}) \end{cases}$$
$$j=\gamma', \cdots, \gamma''\ ;$$

the coefficients are all holomorphic and bounded functions of (x, ε) for

$$(9.9)\qquad |x| < a',\ 0 < |\varepsilon| < b',\ |\arg \varepsilon| < \theta'\ ,$$

admitting uniformlly asymptotic expansions in powers of ε when ε tends to zero in the domain (9.9).

§ 10. Convergence of the formal power series (9.4)~(9.8) appearing in (9.2) and (9.3).

The purpose of this section is to prove that:

All coefficients in (9.2) and (9.3) are holomorphic in the domain

$$(10.1)\qquad |x| < a^0,\ 0 < |\varepsilon| < b^0,\ |\arg \varepsilon| < o^0,\ \max_{l=n'}^{n''} |z_l| < c^0\ .$$

In other words, "the power series appearing on the right hand sides of (9.4)~(9.8) are uniformly convergent in the domain (10.1)"

In the first place, we consider ϕ_j. Let $Z_{n'}, \cdots, Z_{n''}$ be a solution of $(9.2)_2$. Since Z is independent of x, if put $Z_{n'} = z_{n'}^0, \cdots, Z_{n''} = z_{n''}^0$ in (9.4), then the power series (9.4) coincides with the power series of $z_{n'}^0, \cdots, z_{n''}^0$

which represent the j-th component of $\phi^{(2)}$ in Theorem C. Hence, if a^0, b^0, θ^0 and c^0 are suitably chosen, such a power series in $z^0_{n'}, \cdots, z^0_{n''}$ is uniformly convergent. Therefore, the power series appearing on the right hand side of (9.4) represents a holomorphic and bounded function $\phi_j(x, \varepsilon, z)$ in the domain (10.1).

Next we consider the formal power series (9.5) and (9.7). We shall first show that these formal power series are formal solutions of certain differential equations. To do this, differentiating (9.3) term by term, and substituting (7.1) and (9.2) into it, we obtain formal power series of $z_{\gamma'}, \cdots, z_{\gamma''}$. Equating the coefficients of z_h and of $z_{\gamma'}^{l_{\gamma''}} \cdots z_{\gamma''}^{l_{\gamma''}} (l_{\gamma'} + \cdots + l_{''} \geqq 2)$ on both sides of these equalities respectively, we can easily derive the following relations:

$$(10.2) \qquad \varepsilon^{\sigma_\gamma} \frac{d\phi_{jh}}{dx} = \varepsilon^{\sigma_\gamma - \sigma_j} \sum_{l=1}^{n} a_{jl}(x, \varepsilon)\phi_{lh} - \sum_{l=\gamma'}^{\gamma''} \phi_{jl} a_{lh}(x, \varepsilon) - \sum_{l=\gamma'}^{\gamma''} \{\phi_{jl} + \delta_{jl}\} c_{lh}$$

$$+ \varepsilon^{\sigma_\gamma - \sigma_j} \sum_{l=1}^{n} \left\{ (\delta_{lh} + \phi_{lh}) \left[\frac{\partial}{\partial y_l} f_j(x, \varepsilon, y) \right]_{y=\phi(x,\varepsilon,Z)} \right\}$$

$$j = 1, 2, \cdots, n, \ h = \gamma', \cdots, \gamma''$$

and

$$(10.3) \qquad \varepsilon^{\sigma_\gamma} \frac{d\phi_{j\mathfrak{L}_\gamma}}{dx} = \varepsilon^{\sigma_\gamma - \sigma_j} \sum_{l=1}^{n} a_{jl}(x, \varepsilon)\phi_{l\mathfrak{L}_\gamma} - \sum [\delta_{jh} + \phi_{jh}(x, \varepsilon, Z)] c_{h\mathfrak{L}_\gamma}$$

$$- \left(\sum_{i=\gamma'}^{\gamma''} l_i c_{ii}(x, \varepsilon, Z) \right) \phi_{j\mathfrak{L}_\gamma} - \sum [(l_i + 1) \sum (a_{kh} + c_{kh}) \phi_{j\mathfrak{L}_\gamma + e_i - e_h}]$$

$$+ Q_{j\mathfrak{L}_\gamma}(x, \varepsilon, \phi_{\mathfrak{h}_\gamma}(x, \varepsilon, Z), f_{\mathfrak{m}}(x, \varepsilon))$$

$$\mathfrak{L}_\gamma + e_i - e_h = (l_{\gamma'} + \delta_{\gamma' i} - \delta_{\gamma' h}, \cdots, l_{\gamma''} + \delta_{\gamma'' i} - \delta_{\gamma'' h})$$

where $Q_{j\mathfrak{L}_\gamma}(x, \varepsilon, u_{\mathfrak{h}_\gamma}, v_{\mathfrak{m}})$ is a linear form of $v_{\mathfrak{m}}$ $(m_{\gamma'} + \cdots + m_{\gamma''} \leqq l_{\gamma'} + \cdots + l_{\gamma''})$ whose coefficients are polynomials of $u_{\tilde{\mathfrak{h}}_\gamma}$ $(h_{\gamma'} + \cdots + h_{\gamma''} < l_{\gamma'} + \cdots l_{\gamma''})$. Then it is clear that the formal power series (9.5) and (9.6) formally satisfy the differential equations (10.2) and (10.3) respectively if we replace $c(x, \varepsilon, Z)$ by the formal power series given by (9.7) or (9.8).

§ 11. Determination of the functions $\phi_{jh}(x, \varepsilon, z)$ and $c_{jh}(x, \varepsilon, z)$.

1°. Refinement of the differential equations which define the functions ϕ_{jh} and c_{jh}. We put

$$(11.1) \qquad \begin{aligned} &c_{jh}(x, \varepsilon, z) \equiv 0, \quad \text{if} \quad \lambda_j \neq \lambda_h (j, h = \gamma', \cdots, \gamma'') \\ &\phi_{jh}(x, \varepsilon, z) \equiv 0, \quad \text{if} \quad \lambda_j = \lambda_h \text{ and } \sigma_j = \sigma_h \\ &\qquad\qquad (j = 1, 2, \cdots, n, \ h = \gamma', \cdots, \gamma'') \,. \end{aligned}$$

Then, for (j, h) such that $\lambda_j = \lambda_h$ and $\sigma_j = \sigma_h$, we can derive the following equations from (10.2):

$$c_{jh}=\left[\sum_{l=1}^{n}\phi_{lh}\frac{\partial}{\partial y_l}f_j(x,\varepsilon,y)+\frac{\partial}{\partial y_h}f_j(x,\varepsilon,y)\right]_{y=\phi(x,\varepsilon,Z)}\,. \tag{11.2}$$

In fact, to obtain these relations, it is sufficient to show $\sum a_{jl}\phi_{lh}\equiv 0$ and $\sum\phi_{jl}(a_{lh}+c_{lh})\equiv 0$. If we assume the contrary, then $a_{jl}\not\equiv 0$ (or $\phi_{jl}\not\equiv 0$) will imply $\lambda_j=\lambda_l$, $\sigma_j=\sigma_l$ (or $\lambda_j\neq\lambda_l$ or $\sigma_j\neq\sigma_h$), while, if $\phi_{lh}\not\equiv 0$ (or $a_{lh}+c_{lh}\not\equiv 0$), we have $\lambda_l\neq\lambda_h$ and $\sigma_l=\sigma_h$ or $\sigma_l\neq\sigma_h$ (or $\lambda_l=\lambda_h$ and $\sigma_l=\sigma_h$). this we can conclued that $\{\lambda_j\neq\lambda_h$ and $\sigma_j=\sigma_h\}$ or $\{\sigma_j\neq\sigma_h\}$, which is a contradiction.

Substituting the expressions (11.2) into the right hand members of (10.2), we have a self-contained system of the non-linear differential equations

$$\begin{aligned}\varepsilon^{\sigma_\gamma}\frac{d\phi_{jh}}{dx}&=\varepsilon^{\sigma_\gamma-\sigma_j}\sum_{l=1}^{n}a_{jl}(x,\varepsilon)\phi_{lh}-\sum_{\substack{\lambda_l=\lambda_h\\ \sigma_l=\sigma_h}}\phi_{jl}a_{lh}(x,\varepsilon)\\&\quad-\sum_{l=\gamma'}^{\gamma''}\{\phi_{jl}+\delta_{jl}\}\left[\sum_{i=1}^{n}\{\phi_{ih}+\sigma_{ih}\}\frac{\partial}{\partial y_i}f_l\right]_{y=\phi}\\&\quad+\varepsilon^{\sigma_\gamma-\sigma_j}\sum_{l=1}^{n}\left\{(\delta_{lh}+\phi_{lh})\left[\frac{\partial}{\partial y_l}f_j(x,\varepsilon,y)\right]_{y=\phi}\right\}\end{aligned} \tag{11.3}$$

which define the functions $\phi_{jh}(x,\varepsilon,z)$ for $\lambda_j\neq\lambda_h$, $\sigma_j=\sigma_h$ or $\sigma_j\neq\sigma_h$. Multiplying both sides of the equations $(11.3)_{jh}$ by 1 or $\varepsilon^{\sigma_j-\sigma_\gamma}$ according as $\sigma_j\leqq\sigma_\gamma$ or $\sigma_j>\sigma_\gamma$, we obtain

$$\begin{cases}\varepsilon^{\sigma_\gamma}\dfrac{d\phi_{jh}}{dx}=-\displaystyle\sum_{l=\gamma'}^{\gamma''}\phi_{jl}a_{lh}(x,\varepsilon)+H_{jh}[x,\varepsilon,\phi,\phi_{kh}] & (\sigma_j<\sigma_\gamma)\,,\\ \varepsilon^{\sigma_\gamma}\dfrac{d\phi_{jh}}{dx}=\displaystyle\sum_{l=1}^{n}a_{jl}(x,\varepsilon)\phi_{lh}-\sum\phi_{jl}a_{lh}(x,\varepsilon)+H_{jh}[x,\varepsilon,\phi,\phi_{kh}] & (\sigma_j=\sigma_\gamma)\,,\\ \varepsilon^{\sigma_j}\dfrac{d\phi_{jh}}{dx}=\sum a_{jl}(x,\varepsilon)\phi_{lh}+H_{jh}[x,\varepsilon,\phi,\phi_{kh}] & (\sigma_j>\sigma_\gamma)\end{cases} \tag{11.4}$$

where $H_{jh}[x,\varepsilon,\phi,\phi_{kh}]$ are polynomials of ϕ_{kh} ($k=1,2,\cdots,n$, $h=\gamma',\cdots,\gamma''$) whose coefficients are holomorphic and bounded functions of (x,ε) without constant terms (i.e. $\lim H_{jh}[0,\varepsilon,0,\phi_{kh}]=0$). In the expression $\phi(x,\varepsilon,Z)$, Z is a constant solution of the reduced form $(9.2)_2$, and hence Z_j ($n'\leqq j\leqq n''$) can be regarded as parameters. Denoting $Z_{n'},\cdots,Z_{n''}$ by $\tau_{n'},\cdots,\tau_{n''}$, and setting

$$H_{jh}[x,\varepsilon,\phi,\phi_{kh}]\equiv\tilde{H}_{jh}[x,\varepsilon,\tau_{n'},\cdots,\tau_{n''},\phi_{kh}]\,,$$

we have

$$\lim_{\varepsilon\to 0}\tilde{H}_{jh}[0,\varepsilon,0,\cdots,0,\phi_{kh}]=0\,.$$

If $a_{lh}(x, \varepsilon)$ appearing on the right hand sides of (11.4) does not vanish identically, then we have $\sigma_l=\sigma_h$ and λ_l $(\equiv \lim a_{ll}(0, \varepsilon))=\lambda_h$. In virtue of our hypotheses, since the matrix formed by all $a_{jl\cdot 0}(0)(\sigma_j>0, \sigma_l>0)$ is nonsingular and of Jordan's canonical form, (11.4) may be written in the form

$$(11.5)\quad \begin{cases} \varepsilon^{\sigma_\gamma}\dfrac{d\phi_{jh}}{dx}=[-\phi_{jh}\cdot a_{hh\cdot 0}(0)-\phi_{jh-\xi}a_{h-1\,h\cdot 0}(0)]+K_{jh}(x, \varepsilon, \tau, \phi_{kh}) & (\sigma_j<\sigma_\gamma) \\ \varepsilon^{\sigma_\gamma}\dfrac{d\phi_{jh}}{dx}=[(a_{jj\cdot 0}(0)-a_{hh\cdot 0}(0))\phi_{jh}+a_{jj+1\cdot 0}(0)\phi_{j+1\,h}-\phi_{jh-1}a_{h-1\,h\cdot 0}(0)] & \\ \qquad\qquad +K_{jh}(x, \varepsilon, \tau, \phi_{kh}) & (\sigma_j=\sigma_\gamma)\,, \\ \varepsilon^{\sigma_\gamma}\dfrac{d\phi_{jh}}{dx}=[a_{jj\cdot 0}(0)\phi_{jh}+a_{jj+1\cdot 0}(0)\phi_{j+1h}]+K_{jh}(x, \varepsilon, \tau, \phi_{kh}) & (\sigma_j>\sigma_\gamma) \end{cases}$$

$$\lim_{\varepsilon} K_{jh}(0, \varepsilon, 0, \phi_{kh})=0.$$

The following facts should be noticed:

i) The right hand sides contain $m+1$ parameters $\varepsilon, z^0_{n'}, \cdots, z^0_{n''}$.

ii) The coefficient of ϕ_{jh} appearing on the right hand side of $(11.5)_{jh}$ differs from zero.

iii) The system (11.5) possesses a formal solution which is formally developable in the power series of $\tau_{n'}, \cdots, \tau_{n''}$ of the form (7.5) (where we put $z_{n'}=\tau_{n'}, \cdots, z_{n''}=\tau_{n''}$).

iv) If we define a suitable order for a set of indices (j, h), the matrix formed by the coefficients of the ϕ_{jh} appearing in the bracket on the right hand sides of (11.5) is nonsingular and of a triangular form.[3]

2°. Lemma concerning the existence of a solution of a system of nonlinear differential equations.

In order to investigate the analytical meaning of the formal solution of the system (11.5), it is sufficient to consider a following system containing $m+1$ parameters $\varepsilon, \tau_1, \cdots, \tau_m$:

$$(11.6)\qquad \Lambda(\varepsilon)\frac{d\vec{y}}{dx}=\vec{a}(x, \varepsilon, \tau)+A(x, \varepsilon, \tau)\vec{y}+\sum_{\mathfrak{L}}\vec{f}_{\mathfrak{L}}(x, \varepsilon, \tau)y^{(\mathfrak{L})}$$

where

i) $\Lambda(\varepsilon)$ is a diagonal matrix whose diagonal elements are $\varepsilon^{\sigma_j}(\sigma_j>0)$;

ii) $\vec{y}$ is an n-dimensional vector;

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3) For example, it is sufficient to define the order in a following way analogous to M. Hukuhara [5];

i) $\sigma_j<\sigma_\gamma$, $(j, h)<(j', h')\Longleftrightarrow\{j<j'\}$ or $\{j=j'$ and $h<h'\}$;

ii) $\sigma_j=\sigma_\gamma$, $(j, h)<(j', h')\Longleftrightarrow\{j-h>j'-h'\}$ or $\{j-h=j'-h'$ and $j>j'\}$;

iii) $\sigma_j>\sigma_\gamma$, $(j, h)<(j', h')\Longleftrightarrow\{h<h'\}$ or $\{h=h'$ and $j>j'\}$;

Then, for $\sigma_j<\sigma_\gamma$, $(j, h-1)<(j, h)$;

for $\sigma_j=\sigma_\gamma$, $(j+1, h)<(j, h-1)<(j, h)$;

for $\sigma_j>\sigma_\gamma$, $(j+1, h)<(j, h)$.

iii) $\vec{a}(x, \varepsilon, \tau)$, $A(x, \varepsilon, \tau)$ and $\vec{f}_{\mathfrak{L}}(x, \varepsilon, \tau)$ are written as

$$\vec{a}(x, \varepsilon, \tau)=\sum_{\mathfrak{K}} \vec{a}_{\mathfrak{K}}(x, \varepsilon)\tau^{(\mathfrak{K})}$$
$$A(x, \varepsilon, \tau)=\sum_{\mathfrak{K}} A_{\mathfrak{K}}(x, \varepsilon)\tau^{(\mathfrak{K})}$$
$$\vec{f}_{\mathfrak{L}}(x, \varepsilon, \tau)=\sum_{\mathfrak{K}} \vec{f}_{\mathfrak{L},\mathfrak{K}}(x, \varepsilon)\tau^{(\mathfrak{K})}$$

in which the power series on the right hand sides are uniformly convergent for

$$(11.7) \qquad |x|<a,\ 0<|\varepsilon|<b,\ |\arg\varepsilon|<\theta,\ \|\tau\|<c;$$

iv) The components of $\vec{a}_{\mathfrak{K}}(x, \varepsilon)$, $A_{\mathfrak{K}}(x, \varepsilon)$ and $\vec{f}_{\mathfrak{L},\mathfrak{K}}(x, \varepsilon)$ are functions holomorphic and bounded in (x, ε) for

$$(11.8) \qquad |x|<a,\ 0<|\varepsilon|<b,\ |\arg\varepsilon|<\theta\ ,$$

and admit uniformly asymptotic expansions as ε tends to zero in the domain (11.8);

v) Condition S; $\lim\limits_{\varepsilon\to 0} A(0, \varepsilon, 0)$ is nonsingular and of a triangular form.[4)]

Lemma 2. *Suppose that the system* (11.6) *possesses a formal solution*[5)]

$$(11.9) \qquad \vec{y}\sim\sum \vec{p}_{\mathfrak{K}}(x, \varepsilon)\tau^{(\mathfrak{K})}$$

in which the coefficients $\vec{p}_{\mathfrak{K}}(x, \varepsilon)$ are holomorphic and bounded vector functions of (x, ε) in the domain (11.8) *admitting uniformly asymptotic expansions in powers of ε and $\lim\limits_{\varepsilon\to 0} \vec{p}_{\mathfrak{K}}(0, \varepsilon)=0$. Then, there exists a solution $\vec{y}=\vec{\psi}(x, \varepsilon, \tau_1, \cdots, \tau_m)$ holomorphic and bounded in (x, ε, τ) for*

$$(11.10) \qquad |x|<a',\ 0<|\varepsilon|<b',\ |\arg\varepsilon|<\theta',\ \|\tau\|<c'\ ,$$

developable in a uniformly convergent power series of $\tau_1, \cdots, \tau_m$ of the form (11.9), *where the positive constants a', b', θ' and c' must be suitably chosen. Accordingly, the formal series* (11.9) *converge in this domain and represent an actual solution.*

Since the proof is essentially similar to that of Theorem 1 in [3], we omit the proof.

3°. Application of Lemma 2.

By applying Lemma 2 to (11.5), we see that the power series (9.5)

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4) By a suitable linear substitution, the sum of absolute values of the off-diagonal elements can be chosen as small as we want.

5) Under the conditions i)$\sim$v), we can easily verify the existence of a formal solution of the form (11.9).

represents holomorphic functions of (x, ε, z) in (10.1) if a, b, θ and c are sufficiently small positive numbers.

Substituting the function $\phi_{jh}(x, \varepsilon, z)$ just determined into (11.2), the function $c_{jh}(x, \varepsilon, z)$ for $\lambda_j=\lambda_h$ and $\sigma_j=\sigma_h$ is also determined in the domain (10.1).

Thus we can state the following:

Lemma 3. *The functions* $\phi_{jh}(x, \varepsilon, z)$ *and* $c_{jh}(x, \varepsilon, z)$ *are uniquely determined under the conditions* (11.1). *Hence they satisfy the system* (10.2) (*where we put* $Z=z$) *and are holomorphic functions of* (x, ε, z) *for* (10.1). *Consequently, we can replace the formal equality* $\sim$ *in* (9.5) *and* (9.7) *by the true equality* $=$.

§ 12. Determination of the functions $\phi_{j\mathfrak{L}_\gamma}$ and $c_{j\mathfrak{L}_\gamma}$.

1°. Classification of the set of indices $(j, l_{\gamma'}, \cdots, l_{\gamma''})$. Suppose that the functions $\phi_{j\mathfrak{K}_\gamma}(x, \varepsilon, z)$ and $c_{j\mathfrak{K}_\gamma}(x, \varepsilon, z)$ $(k_{\gamma'}+\cdots+k_{\gamma''}<N)$ are determined so that they are holomorphic for (10.1), or, what is the same thing, that the power series on the right hand sides of (9.6) and (9.8) are uniformly convergent for (10.1) $(k_{\gamma'}+\cdots+k_{\gamma''}<N)$. Then, as we have already shown, the equations for $\phi_{j\mathfrak{L}_\gamma}(x, \varepsilon, z)$ and $c_{j\mathfrak{L}_\gamma}(x, \varepsilon, z)$ can be written as (10.3). And, the expressions $Q_{j\mathfrak{L}_\gamma}$ in (10.3) are known functions.

For convenience' sake, we adopt a symbol

$$\lambda_{j\mathfrak{L}_\gamma}\equiv\lambda_j-k_{\gamma'}\lambda_{\gamma'}-\cdots-k_{\gamma''}\lambda_{\gamma''} .$$

Let F_N be the totality of the arrangement $(j, \mathfrak{K}_\gamma)$ such that $1\leqq j\leqq n$ and $k_{\gamma'}+\cdots+k_{\gamma''}=N$. We divide F_N into classes $(j, \mathfrak{L}_\gamma)_N^1, \cdots, (j, \mathfrak{L}_\gamma)_N^H$ according to the values of $\lambda_{j\mathfrak{L}_\gamma}$, i.e. $(j, \mathfrak{L}_\gamma)$ and $(l, \mathfrak{K}_\gamma)$ belong to the same class if and only if $\lambda_{j\mathfrak{L}_\gamma}=\lambda_{l\mathfrak{K}_\gamma}$. Then, we assert that, in the system (10.3), the indices of undetermined ϕ's appearing on both sides of the equation all belong to the same class. Indeed, such undetermined ϕ's are divided into following three groups:

(1) $\phi_{j\mathfrak{L}_\gamma}$

(2) $\phi_{l\mathfrak{L}_\gamma}$ is the term $\sum a_{jl}(x, \varepsilon)\phi_{l\mathfrak{L}_\gamma}$

(3) $\phi_{j\mathfrak{L}_\gamma+e_i-e_h}$ in the term $\sum_{h\neq i}(a_{ih}(x, \varepsilon, z)+c_{ih}(x, \varepsilon, z))\phi_{j\mathfrak{L}_\gamma+e_i-e_h}$

To the group (2) belongs ϕ's such that $a_{jl}(x, \varepsilon)\not\equiv 0$. Since, by (7.3), the condition $a_{jl}(x, \varepsilon)\not\equiv 0$ implies $\sigma_j=\sigma_l$ and $\lambda_j=\lambda_l$, we have

$$\lambda_{l\mathfrak{L}_\gamma}-\lambda_{j\mathfrak{L}_\gamma}=\lambda_l-\lambda_j=0 .$$

Therefore, every $(l, \mathfrak{L}_\gamma)$ in the group (2) belongs to the same class as $(j, \mathfrak{L}_\gamma)$.

For indices $(j, \mathfrak{L}_\gamma)$ and $(j, \mathfrak{L}_\gamma+e_i-e_h)$ in the group (3), we obtain

$$\lambda_{j\mathfrak{L}_\gamma+e_i-e_h}-\lambda_{j\mathfrak{L}_\gamma}=\lambda_h-\lambda_i=0\ ,$$

since the condition $a_{ih}(x, \varepsilon)+c_{ih}(x, \varepsilon, z)\not\equiv 0$ implies $\lambda_h=\lambda_i$ and $\sigma_h=\sigma_i$. Therefore, $(j, \mathfrak{L}_\gamma+e_i-e_h)$ and $(j, \mathfrak{L}_\gamma)$ belong to the same class.

2°. Refinement of the differential equations which define the functions $\phi_{j\mathfrak{L}_\gamma}$ and $c_{j\mathfrak{L}_\gamma}$.

Now our purpose is to determine the functions $\phi_{j\mathfrak{L}_\gamma}(x, \varepsilon, z)$ and $c_{j\mathfrak{L}_\gamma}(x, \varepsilon, z)$ from (10.3) under the conditions;

$$\begin{cases}\phi_{j\mathfrak{L}_\gamma}(x, \varepsilon, z)\equiv 0, & \text{if} \quad \lambda_{j\mathfrak{L}_\gamma}=0 \quad \text{and} \quad \sigma_j=\sigma_\gamma\ , \\ c_{j\mathfrak{L}_\gamma}(x, \varepsilon, z)\equiv 0, & \text{if} \quad \lambda_{j\mathfrak{L}_\gamma}\neq 0 \quad \text{and} \quad \sigma_j\neq\sigma_\gamma\ .\end{cases} \tag{12.1}$$

We shall first consider the indices $(j, \mathfrak{L}_\gamma)$ such that $\lambda_{j\mathfrak{L}_\gamma}=0$ and $\sigma_j=\sigma_\gamma$. As we have remarked above, the indices of undetermined ϕ's in this equation all belong to the same class. For in this case, we can easily reduce $(10.3)_{j\mathfrak{L}_\gamma}$ to the form (where we put $z=Z$)

$$c_{j\mathfrak{L}_\gamma}=Q_{j\mathfrak{L}_\gamma}(x, \varepsilon, z)\ . \tag{12.2}$$

Thus $c_{j\mathfrak{L}_\gamma}(x, \varepsilon, z)$ is a holomorphic function of (x, ε, z) in the domain (10.1).

Next we consider the class of indices $(j, \mathfrak{L}_\gamma)$ such that $\sigma_j>\sigma_\gamma$.

Substituting (12.2) into (11.3), the equations (11.3) multiplied by $\varepsilon^{\sigma_j-\sigma_\gamma}$, will be reduced to

$$\begin{aligned}\varepsilon^{\sigma_j}\frac{d\phi_{j\mathfrak{L}_\gamma}}{dx}=&\sum_{\lambda_j=\lambda_l} a_{jl}(x, \varepsilon)\phi_{l\mathfrak{L}_\gamma}-\varepsilon^{\sigma_j-\sigma_\gamma}\Big(\sum_{l=\gamma'}^{\gamma''} k_l(a_{ll}+c_{ll})\Big)\phi_{l\mathfrak{L}_\gamma} \\ &-\varepsilon^{\sigma_j-\sigma_\gamma}\sum_{l=\gamma'}^{\gamma''}\Big[(k_l+1)\sum_{h\neq l}(a_{lh}+c_{lh})\phi_{j\mathfrak{L}_\gamma+e_l-e_h}\Big] \\ &+\varepsilon^{\sigma_j-\sigma_\gamma}R_{j\mathfrak{L}_\gamma}(x, \varepsilon, z)\end{aligned} \tag{12.3}$$

in which $R_{j\mathfrak{L}_\gamma}$ are holomorphic and bounded functions for (10.1). Obviously, this system possesses a formal solution (9.6) (where we put $z=Z$) developable formally in power series of $z_{n'}, \cdots, z_{n''}$. Since, as was already remarked, $z_{n'}, \cdots, z_{n''}$ are regarded as parameters on which the system depends holomorphically for $\max\limits_{l=n'}^{n''}|z_l|<c^0$, these equations will be regarded as a form of the system (11.6). So we can conclude from Lemma 2 that (12.3) possesses a solution $\phi_{j\mathfrak{L}_\gamma}=\phi_{j\mathfrak{L}_\gamma}(x, \varepsilon, Z)$ where $\phi_{j\mathfrak{L}_\gamma}(x, \varepsilon, z)$ are holomorphic functions of (x, ε, z) in the domain (10.1). In other words, in the expressions of the formal solution (9.6), the power series converge uniformly for (10.1).

i.e. we can replace the formal equality $\sim$ in (9.6) by the true equality $=$.

Finally, we have to consider the class of indices $(j, \mathfrak{L}_\gamma)$ for which $\sigma_j\leqq\sigma_\gamma$. We shall define the order to the elements $(j, \mathfrak{L}_\gamma)$ of this class in a following way.

If $j<k$

or

$$j=k \quad \text{and} \quad l_{\gamma'}=k_{\gamma'}, \cdots, l_\gamma=k_\gamma, l_{\gamma+1}<k_{\gamma+1}, (\gamma+1\leqq\gamma'')$$

we shall call that $(j, \mathfrak{L}_\gamma)$ precedes $(k, \mathfrak{K}_\gamma)$ and denote it by $(j, \mathfrak{L}_\gamma)\prec(k, \mathfrak{K}_\gamma)$. In virtue of the order just defined above and $c_{l\mathfrak{K}_\gamma}(x, \varepsilon, z)\equiv 0$, it follows that the matrix formed by the constant coefficients of undtermined ϕ's is nonsingular and of a triangular form. Consequently, this case will be reduced to the precedent one.

Thus we have the following:

Lemma 4. *The coefficients* $c_{j\mathfrak{L}_\gamma}(x, \varepsilon, z)$ *and* $\phi_{j\mathfrak{L}_\gamma}(x, \varepsilon, z)$ *are uniquely determined one after another, under the condition* (12.1), *so that by putting* $z=Z$, $\phi_{j\mathfrak{L}_\gamma}(x, \varepsilon, z)$ *represent a solution* (*of the system* (10.3)) *holomorphic and bounded in* (x, ε, z) *for* (10.1) *if we replace* $c_{j\mathfrak{L}_\gamma}$ *by the function* $c_{j\mathfrak{L}_\gamma}(x, \varepsilon, z)$.

§ 13. Formal solutions and a reduced system.

Since the coefficients in (9.2) are holomorphic function owing to the Lemmas 3 and 4 and (9.2) is a polynomial of $z_{\gamma'}, \cdots, z_{\gamma''}$, the reduced system (9.2) possesses a well-defined meaning. Let $z_j=U_j(x, \varepsilon, z^0_{n'}, \cdots, z^0_{n''}; x_0, z^0)(j=\kappa', \cdots, \kappa'')$ be a holomorphic solution of the reduced system (9.2) satisfying the initial condition

$$z^0_j=U_j(x_0, \varepsilon, z^0_{n'}, \cdots, z^0_{n''}; x_0, z^0)\,.$$

Clearly $U_j(x, \varepsilon, z^0_{n'}, \cdots, z^0_{n''}; x_0, z^0)$, $j=n', \cdots, n''$, are independent of x.

Thus we have established the following.

Theorem 2. *The differential equations* (7.1) *possess formal solutions of the form*:

$$(F_+)\qquad y_j\sim\phi_j(x, \varepsilon, U_{n'}, \cdots, U_{n''})+\sum\{\delta_{jl}+\phi_{jl}(x, \varepsilon, U_{n'}, \cdots, U_{n''})\}U_l$$
$$+\sum_{|\mathfrak{L}_1|\geqq 2}\phi_{j\mathfrak{L}_1}(x, \varepsilon, U_{n'}, \cdots, U_{n''})U_\gamma^{l_1}\cdots U_\alpha^{l_\alpha}\,,\qquad j=1, 2, \cdots, n,$$

$$(F_+)\qquad y_j\sim\phi_j(x, \varepsilon, U_{n'}, \cdots, U_{n''})+\sum\{\delta_{jl}+\phi_{jl}(x, \varepsilon, U_{n'}, \cdots, U_{n''})\}U_l$$
$$+\sum_{|\mathfrak{L}_\beta|\geqq 2}\phi_{j\mathfrak{L}_\beta}(x, \varepsilon, U_{n'}, \cdots, U_{n''})U_\beta^{l_\beta}\cdots U_n^{l_n}\,,\qquad j=1, 2, \cdots, n,$$

where

(1) $\phi_j(x, \varepsilon, z)$, $\phi_{jl}(x, \varepsilon, z)$ *and* $\phi_{j\mathfrak{L}_\gamma}(x, \varepsilon, z)$ *are holomorphic and bounded in* (x, ε, z) *for*

$$(13.1)\qquad |x|<a_1,\ 0<|\varepsilon|<b_1,\ |\arg\varepsilon|<\theta_1,\ \|z\|<c_1\,,$$

and, especially, $\phi_j(x, \varepsilon, U)$ *is an actual solution of* (7.1) *which coincides*

with the solution $\vec{\phi}^{(2)}$ *stated in Theorem C.*
(2) $U_{\gamma'}, \cdots, U_{\gamma''}, U_{n'}, \cdots, U_{n''}$ *is a holomorphic solution of the reduced system*

$$\varepsilon^{\sigma_j}\frac{dz_j}{dx}=\begin{cases}\sum_{h=\gamma'}^{\gamma''}\{a_{jh}(x,\varepsilon)+c_{jh}(x,\varepsilon,z_{n'},\cdots,z_{n''})\}z_h \\ \quad +\sum c_{j\mathfrak{L}_\gamma}(x,\varepsilon,z_{n'},\cdots,z_{n''})z_{\gamma'}^{l\gamma'}\cdots z_{\gamma''}^{l\gamma''}, & j=\gamma',\cdots,\gamma'', \\ 0 & j=n',\cdots,n'', \end{cases} \tag{13.2}$$

satisfying the initial condition

$$z_j^0=U_j(x_0,\varepsilon,z_{n'}^0,\cdots,z_{n''}^0;x_0,z^0)$$

in which the right hand sides of (13.2) *are polynomials in* $z_{\gamma'},\cdots,z_{\gamma''}$ *whose coefficients are holomorphic functions of* $(x,\varepsilon,z_{n'},\cdots,z_{n''})$ *in the domain* (13.1).

Remark. It is clear that we have $U_j\equiv z_j$ for $j=n',\cdots,n''$. Therefore, if we put $z_{n'}^0=\cdots=z_{n''}^0=0$ we have $U_{n'}=\cdots=U_{n''}=0$ and consequently, $U_1,\cdots,U_\alpha$ or $U_\beta,\cdots,U_n$ are reduced to the solution $Z_1,\cdots,Z_\alpha$ or $Z_\beta,\cdots,Z_n$ of the system (4.8). On the other hand, if we put $z_1^0=\cdots=z_\alpha^0=0$ or $z_\beta^0=\cdots=z_n^0=0$, we have $U_1=\cdots=U_\alpha=0$ or $U_\beta=\cdots=U_n=0$.

IV. Auxiliary lemmas

§14. The property of $\max_{l=\gamma'}^{\gamma''}|U_l|^{1/\mu_l}$.

In virtue of the inequalities (4.2), if $\theta^0(>0)$ is chosen sufficiently small, we have

$$\begin{aligned} &\max|\arg\lambda_j-\Omega-\sigma_j\arg\varepsilon|\leqq\frac{\pi}{2}-2\theta^0 && j<n' \\ &\max|\arg\lambda_j-\Omega-\sigma_j\arg\varepsilon-\pi|\leqq\frac{\pi}{2}-2\theta^0 && j>n'' \end{aligned} \tag{14.1}$$

for $|\arg\varepsilon|\leqq\theta^0$.

Denote by $\vartheta(\rho,\theta^0,\Omega)$ an inner part of the lozenge whose four vertices are

$$a^{(1)}=\rho e^{-i\Omega},\ a^{(2)}=ia^{(1)}\cdot\tan\theta^0,\ a^{(3)}=-a^{(1)},\ a^{(4)}=-a^{(2)}$$

respectively and define a point x_γ^* by

$$x_\gamma^*=a^{(3)} \qquad (\gamma',\cdots,\gamma'')=(1,\cdots,\alpha)$$

or

$$x_\gamma^*=a^{(1)} \qquad (\gamma',\cdots,\gamma'')=(\beta,\cdots,n)$$

where ρ is a sufficiently small number such that $0<\rho\leqq a$.

Let $\Gamma_{\gamma x_0}$ be the segment joining x_0 and x_γ^* where x_0 is a arbitrary point in $\vartheta(\rho, \theta^0, \Omega)$. Then the variable point on this segment is expressed as

$$x=x_\gamma^*+s\cdot e^{i\varphi},$$
$$\varphi=\arg(x_0-x_\gamma^*)$$

where s is the length of the segment from x_γ^* to x.

Lemma 5. *Let*

$$\mu_h=\begin{cases} Re[\lambda_h e^{-i\Omega}] & h=1, \cdots, \alpha, \\ -Re[\lambda_h e^{-i\Omega}] & h=\beta, \cdots, n. \end{cases}$$

Then, on $\Gamma_{\gamma x_0}$, *we have*

$$\frac{d}{ds}\left\{\max_{l=\gamma'}^{\gamma''}|U_l|^{1/\mu_l}\right\}\geqq\left\{\frac{\sin\theta^0}{2|\varepsilon|^{\sigma_\gamma}}\min_{l=\gamma'}^{\gamma''}\frac{|\lambda_l|}{\mu_l}\right\}\max_{l=\gamma'}^{\gamma''}|U_l|^{1/\mu_l} \tag{14.2}$$

for

$$0<|\varepsilon|<b'', |\arg\varepsilon|<\theta^0, \|\tau\|<c''.$$

Proof. The functions $c_{j\mathfrak{k}_\gamma}(x, \varepsilon, \ )$ appearing on the right hand sides of (11.4) (we put $z=\tau$) are finite in number. By linear substitution $z_j=K^\gamma u_j$, if necessary, the value of

$$\sum_{l=1}^{n}|a_{jl}(x, \varepsilon)+c_{jl}(x, \varepsilon, \tau)-\lambda_j\delta_{jl}| \qquad (j=1, 2, \cdots, n)$$

can be taken as small as we want.

Therefore, if ρ, b'' and c'' are chosen sufficiently small, the inequality

$$\sum_{l=1}^{n}|a_{jl}(x, \varepsilon)+c_{jl}(x, \varepsilon, \tau)-\lambda_j\delta_{jl}|+\sum_{\mathfrak{k}_\gamma}|c_{j\mathfrak{k}_\gamma}(x, \varepsilon, \tau)|\leqq\frac{|\lambda_j|\sin\theta^0}{2} \tag{14.3}$$

will hold in the domain

$$|x|<\rho, 0<|\varepsilon|<b'', |\arg\varepsilon|<\theta^0, \|\tau\|<c''. \tag{14.4}$$

In the case if $(\gamma', \cdots, \gamma'')=(1, \cdots, \alpha)$, we have

$$-\theta^0<\varphi+\Omega<\theta^0$$

for $\varphi=\arg(x_0-x_\gamma^*)$ where x_0 is an arbitrary point of $\vartheta(\rho, \theta^0, \Omega)$. By virtue of (14.1), we obtain

$$|\arg[\lambda_j e^{i\varphi}\varepsilon^{-\sigma_j}]|\leqq\frac{\pi}{2}-\theta^0 \qquad j<n'$$

and consequently

$$\cos(\arg[\lambda_j e^{i\varphi}\varepsilon^{-\sigma_j}])\geqq\sin\theta^0>0 \qquad j<n' .$$

Therefore we have

$$Re[\lambda_j e^{i\varphi}\varepsilon^{-\sigma_j}]=\frac{|\lambda_j|\sin\theta^0}{|\varepsilon|^{\sigma_j}} \tag{14.5}$$

for $x\in\vartheta(\rho, \theta^0, \Omega)$, $|\arg\varepsilon|\leqq\theta^0$.

If we choose a point x^0 arbitrarily on Γ_{1x_0}, there exists an index h such that

$$\max_{l=1}^{\alpha}|U_l|^{1/\mu_l}=|U_h|^{1/\mu_h} .$$

Naturally h depends upon the choice of x_0. The conditions

$$c_{h\mathfrak{L}_1}(x, \varepsilon, \tau)\neq 0 \quad\text{and}\quad (a_{hl}(x, \varepsilon)+c_{hl}(x, \varepsilon, \tau))\neq 0$$

imply

$$|U_{h+1}|^{l_{h+1}}\cdots|U_\alpha|^{l_\alpha}\leqq|U_h| \quad\text{and}\quad |U_l|\leqq|U_h| \tag{14.6}$$

at the point x^0 respectively. In fact. suppose that $c_{h\mathfrak{L}_1}(x, \varepsilon, \tau)\neq 0$. Then, we must have the relation

$$\mu_h=l_{h+1}\mu_{h+1}+\cdots+l_\alpha\mu_\alpha$$

Thus we obtain

$$\begin{aligned}|U_h|&=(|U_h|^{1/\mu_h})^{l_{h+1}\mu_{h+1}}\cdots(|U_h|^{1/\mu_h})^{l_\alpha\mu_\alpha}\\&\geqq(|U_{h+1}|^{1/\mu_{h+1}})^{l_{h+1}\mu_{h+1}}\cdots(|U_\alpha^{1/\mu_h}|)^{l_\alpha\mu_\alpha}\\&=|U_{h+1}|^{l_{h+1}}\cdots|U_\alpha|^{l_\alpha}\end{aligned}$$

which is the first inequality of (14.6).

The second inequality of (14.6) can be proved similarly.

Now the inequality (14.2) can be proved easily. Let s^0 be the length of the segment $x_\gamma^* x^0$. Then

$$\begin{aligned}\Big[\frac{d}{ds}\{\max|U_l|^{1/\mu_l}\}\Big]_{s=s^0}&=\Big[\frac{d}{ds}|U_h|^{1/\mu_h}\Big]_{s=s^0}=\Big(\frac{1}{\mu_h}|U_h|^{1/\mu_h}\frac{1}{|U_h|}\frac{d}{ds}|U_h|\Big)_{s=s^0}\\&=\frac{1}{\mu_h}\Big[\max|U_l|^{1/\mu_l}Re\Big(\frac{d}{ds}\log U_h\Big)\Big]_{s=s^0}=\frac{1}{\mu_h}\Big[\max|U_l|^{1/\mu_l}Re\Big(\frac{1}{U_h}\frac{dU_h}{dx}e^{i\varphi}\Big)\Big]_{x=x^0}\end{aligned}$$

Making use of the equation $(11.4)_h$, we have

$$Re\Big(\frac{1}{U_h}\frac{dU_h}{dx}e^{i\varphi}\Big)\geqq Re\Big(\frac{\lambda_h}{\varepsilon^{\sigma_1}}e^{i\varphi}\Big)-\frac{1}{|\varepsilon|^{\sigma_1}}\Big\{\sum_l|a_{hl}+c_{hl}-\lambda_h\delta_{hl}|\frac{|U_l|}{|U_h|}+\sum_{\mathfrak{L}_1}|c_{h\mathfrak{L}_1}|\frac{|U_{h+1}^{l_{h+1}}\cdots U_\alpha^{l_\alpha}|}{|U_h|}\Big\}$$

Thus, from (14.3), (14.5) and (14.6)

$$Re\Big(\frac{1}{U_h}\frac{dU_h}{dx}e^{i\varphi}\Big)\geqq\frac{|\lambda_h|\sin\theta^0}{|\varepsilon|^{\sigma_1}}-\frac{|\lambda_h|\sin\theta^0}{2|\varepsilon|^{\sigma_1}}=\frac{|\lambda_h|\sin\theta^0}{2|\varepsilon|^{\sigma_1}}$$

whence immediately follows the required inequality.

As for $(\gamma',\cdots,\gamma'')=(\beta,\cdots,n)$, similar proof is possible. Q.E.D.

§15. Lemma (Fundamental inequality).

Now we shall prove our final lemma.

Lemma 6. *Put*

$$\Lambda_j(x,\varepsilon)\equiv\lambda_j\varepsilon^{\sigma_j}(x-x_\gamma^*)\ .$$

Then, on the segment $\Gamma_{\gamma x_0}$, we have the inequality

$$(15.1)\quad \frac{d}{ds}\Big\{\exp(-Re\Lambda_j(x,\varepsilon))\max_{l=\gamma'}^{\gamma''}|U_l|^{N/\mu_l}\Big\}$$
$$\leqq\begin{cases}\dfrac{N\sin\theta^0}{4|\varepsilon|^{\sigma_\gamma}}\min\limits_{l=\gamma'}^{\gamma''}\dfrac{|\lambda_l|}{\mu_l}\exp(-Re\Lambda_j(x,\varepsilon))\max\limits_{l=\gamma'}^{\gamma''}|U_l|^{N/\mu_l}\,, & (\sigma_j\leqq\sigma_\gamma)\,,\\ \dfrac{|\lambda_j|\sin\theta^0}{4|\varepsilon|^{\sigma_j}}\cdot\exp(-Re\Lambda_j(x,\varepsilon))\max\limits_{l=\gamma'}^{\gamma''}|U_l|^{N/\mu_l}\,, & (\sigma_j>\sigma_\gamma)\end{cases}$$

for $0<|\varepsilon|<b''$, $|\arg\varepsilon|\leqq\theta^0$ and sufficiently large N.

Proof. It follows from Lemma 5 that

$$\frac{d}{ds}\Big\{\exp(-Re\Lambda_j(x,\varepsilon))\max_{l=\gamma'}^{\gamma''}|U_l|^{N/\mu_l}\Big\}$$
$$=\Big[-\max_l|U_l|^{N/\mu_l}\cdot\frac{d}{ds}Re\Lambda_j(x,\varepsilon)+N\Big(\max_l|U_l|^{1/\mu_l}\Big)^{N-1}\frac{d}{ds}\max|U_l|^{1/\mu_l}\Big]\times\exp[-Re\Lambda_j(x,\varepsilon)]$$
$$=\Big[-\frac{d}{ds}Re\Lambda_j(x,\varepsilon)+N\Big(\max_l|U_l|^{1/\mu_l}\Big)^{-1}\cdot\frac{d}{ds}\max_l|U_l|^{1/\mu_l}\Big]\times\exp(-Re\Lambda_j(x,\varepsilon))\max_l|U_l|^{N/\mu_l}$$
$$\geqq\Big[-\frac{d}{ds}Re\Lambda_j(x,\varepsilon+)\frac{N\sin\theta^0}{2|\varepsilon|^{\sigma_\gamma}}\min_l\frac{|\lambda_l|}{\mu_l}\Big]\exp(-Re\Lambda_j(x,\varepsilon))\max|U_l|^{N/\mu_l}\ .$$

If $\sigma_j\leqq\sigma_\gamma$, we have

$$-\frac{d}{ds}Re\,\Lambda_j(x,\varepsilon)+\frac{N\sin\theta^0}{2|\varepsilon|^{\sigma_\gamma}}\min\frac{|\lambda_l|}{\mu_l}$$

by taking N sufficiently large.

If $\sigma_j>\sigma_\gamma$, we have either

$$j>n'' \quad \text{i.e.} \quad (\gamma',\cdots,\gamma'')=(1,\cdots,\alpha)$$

or

$$j<n' \quad \text{i.e.} \quad (\gamma',\cdots,\gamma'')=(\beta,\cdots,n)\,.$$

In these cases, by the same way just as employed in the proof of lemma 5, we have

$$-\frac{d}{ds}Re\,\Lambda_j(x,\varepsilon)=-Re[\lambda_j\varepsilon^{-\sigma_j}e^{i\varphi}]\geqq\frac{|\lambda_j|\sin\theta^0}{|\varepsilon|^{\sigma_j}}$$

Consequently, for sufficiently large N, it holds

$$-\frac{d}{ds}Re\,\Lambda_j(x,\varepsilon)+\frac{N\sin\theta^0}{2|\varepsilon|^{\sigma_\gamma}}\min_{l=\gamma'}^{\gamma''}\frac{|\lambda_l|}{\mu_l}>\frac{|\lambda_j|\sin\theta^0}{4|\varepsilon|^{\sigma_\gamma}}\,. \qquad \text{Q.E.D.}$$

V. Main theorem

§ 16. Main theorem.

The principal problem of this work may be phrased as follows:

Theorem 3. *In the expressions of the formal solutions* (F_+) *and* (F_-), *the formal power series in their right hand members converge uniformly in the domain*

$$(16.1) \qquad |x|<\alpha^0,\ 0<|\varepsilon|<\beta^0,\ |\arg\varepsilon|<\theta^0,\ \max_{l=\kappa'}^{\kappa''}|U_l|^{1/\mu_l}<\zeta^0$$

where $\alpha^0, \beta^0, \theta^0$ *and* ζ^0 *are sufficiently small positive constants and*

$$\mu_j=\begin{cases}Re[\lambda_je^{-i\Omega}] & j=1,\cdots,\alpha\,,\\ 1 & j=n',\cdots,n''\,,\\ -Re[\lambda_je^{-i\Omega}] & j=\beta,\cdots,n\,.\end{cases}$$

In other words, formal solutions (F_+) and (F_-) represent actual solution of the system (7.1).

Corrollary 1. (i) *If we put* $z^0_{n'}=\cdots=z^0_{n''}=0$, *then the solutions* (F_+) *and* (F_-) *coincide with the solutions* $(F^{(1)})$ *and* $(F^{(3)})$ *in Theorem C respectively.*

(ii) *If* $z_1^0=\cdots=z_\alpha^0=0$ *and* $z_\beta^0=\cdots=z_n^0=0$, *then the solutions* (F_+) *and* (F_-) *coincide with the solution* $(F^{(2)})$.

Corrollary 2. *A sufficient condition that the system* (1.1) *possesses a general solution is that the following case holds*:

$$\text{(i)} \qquad \alpha+1=n' \text{ and } n''=n\begin{pmatrix} \textit{i.e. } \sigma_1=\cdots=\sigma_\alpha>0, \sigma_{\alpha+1}=\cdots=\sigma_n=0 \\ \textit{and } \lambda_1 e^{-i\Omega}\geqq\cdots\geqq\lambda_\alpha e^{-i\Omega}>0 \end{pmatrix}$$

or the same thing

$$\text{(ii)} \qquad n'=1 \quad \text{and} \quad n''+1=\beta\,.$$

§ 17. Proof of the theorem.

We will prove the convergence of the formal solution (F_+). As regards the formal solution (F_-), the discussion is carried out in a similar way.

(1°) Preliminary transformation. Let us put $z_{n'}^0=\tau_{n'}, \cdots, z_{n''}^0=\tau_{n''}$ and define

$$W_{jN}(x, \varepsilon, \tau, z)=\delta_j z_j+\sum_{k_1\mu_1+\cdots+k_\alpha\mu_\alpha<N} \phi_{j\mathfrak{K}_1}(x, \varepsilon, \tau) z_1^{k_1}\cdots z_\alpha^{k_\alpha} \tag{16.2}$$

where

$$\delta_j=\begin{cases} 1 & j=1, \cdots, \alpha\,, \\ 0 & j\neq 1, \cdots, \alpha\,. \end{cases}$$

Since $W_{jN}(x, \varepsilon, \tau, z)$ are polynomials in $z_1, \cdots, z_\alpha$ with bounded coefficients and $\lim_{\varepsilon\to 0} W_{jN}(0, \varepsilon, 0, 0)=0$, there exist small positive numbers $\alpha', \beta', \zeta', \xi'$ and ξ^0 such that the inequalities

$$|W_{jN}(x, \varepsilon, \tau, z)|+\xi'<\xi^0 \qquad j=1, 2, \cdots, n$$

hold on the domain

$$|x|<\alpha', 0<|\varepsilon|<\beta', |\arg\varepsilon|<\theta^0, \|\tau\|<\zeta', \max_{l=1}^{\alpha}|z_l|^{1/\mu_l}<\zeta'$$

and, moreover, the right hand sides of (7.1) are holomorphic functions of (x, ε, y) in the domain

$$|x|<\alpha', 0<|\varepsilon|<\beta', |\arg\varepsilon|<\varepsilon^0, \|y\|<\xi^0\,.$$

Now we make a transformation

$$y_j=v_j+W_{jN}(x, \varepsilon, \tau, U(x, \varepsilon, \tau, x_0, z^0)) \qquad j=1, 2, \cdots, n\,, \tag{16.3}$$

and let

$$(16.4)\qquad \varepsilon^{\sigma_j}\frac{dv_j}{dx}=\lambda_j v_j+g_j(x,\varepsilon,\tau,U_1,\cdots,U_\alpha,v_1,\cdots,v_n)$$

be the transformed system. As is easily verified,

$$g_j(x,\varepsilon,\tau,u,v)=\sum_{l=1}^{n}(a_{jl}(x,\varepsilon)-\lambda_j\delta_{jl})v_l+\sum a_{jl}(x,\varepsilon)W_{lN}(x,\varepsilon,\tau,u)$$
$$+f_j(x,\varepsilon,\tau,v+W_N(x,\varepsilon,\tau,u))-\varepsilon^{\sigma_j}\frac{\partial W_{jN}}{\partial x}-\varepsilon^{\sigma_j-\sigma_1}\sum_{l=1}^{\alpha}\frac{\partial W_{jN}}{\partial u_l}\cdot\varepsilon^{\sigma_1}\frac{du_l}{dx},$$

where the expressions $\varepsilon^{\sigma_1}du_l/dx$ should be replaced by the second members of (13.2).

By a simple consideration, we see that $g_j(x,\varepsilon,\tau,u,v)$ are holomorphic and bounded functions of (x,ε,τ,v) in the domain

$$(16.5)\qquad |x|<\alpha',\ 0<|\varepsilon|<\beta',\ |\arg\varepsilon|<\theta^0,\ \|\tau\|<\zeta',\ \|v\|<\xi'.$$

On the other hand, the system (16.4) evidently admits a formal solution of the form

$$(16.6)\qquad v_j\sim\sum_{l_1\mu_1+\cdots+l_\alpha\mu_\alpha\geqq N}\varphi_{j\mathfrak{L}_1}(x,\varepsilon,\tau)U_1^{l_1}\cdots U_\alpha^{l_\alpha}\qquad j=1,2,\cdots,n.$$

So we can easily verify that the positive constants L and B_N can be so chosen that

$$(16.7)\qquad |g_j(x,\varepsilon,\tau,u,v)|\leqq A_j\|v\|+B_N|\varepsilon|^{\sigma_j^*}\max_{l=1}^{\alpha}|u_l|^{N/\mu_l}$$

$$(16.8)\qquad |g_j(x,\varepsilon,\tau,u,v)-g_j(x,\varepsilon,\tau,u,w)|\leqq A_j\|v-w\|$$

$$(16.9)\qquad A_j=|a_{jj-1,0}(0)|+L(\alpha'+\beta'+\zeta')$$

$$(16.10)\qquad \sigma_j^*=\min\{\sigma_j-\sigma_1,0\}$$

Next we make a substitution

$$v=\tilde{w}_j e^{\Lambda_j(x,\varepsilon)}$$

where $\Lambda_j(x,\varepsilon)$ defined in Lemma 6.

Then, (16.4) will be reduce to

$$(16.11)\qquad \frac{d\tilde{w}_j}{dx}=\varepsilon^{-\sigma_j}g_j(x,\varepsilon,\tau,U(x,\varepsilon,\tau,x_0,z^0),\tilde{w}e^{\Lambda(x,\varepsilon)}))e^{-\Lambda_j(x,\varepsilon)}.$$

(2°) A family F and a mapping $\mathfrak{A}$.

Let F be the family of the systems $\{\varphi_1(x,\varepsilon,\tau,u),\cdots,\varphi_n(x,\varepsilon,\tau,u)\}$, where $\varphi_j(x,\varepsilon,\tau,u)$ are holomorphic and bounded functions of (x,ε,τ,u) in the domain

(16.12) $$x \in \vartheta(\rho, \theta^0, \Omega),\ 0<|\varepsilon|<\beta',\ |\arg \varepsilon|<\theta^0,\ \|\tau\|<\zeta^0,\ \max|u_l|^{1/\mu_l}<\zeta^0$$

and satisfying the inequalities

(16.13) $$|\varphi_j(x, \varepsilon, \tau, u)| \leqq K_N e^{-Re A_j(x,\varepsilon)} \max_{l=1}^{\alpha} |u_l|^{N/\mu_l}$$

where $0<\rho^0 \leqq \rho'$ and $0<K_N(\zeta^0)^N<\zeta'$. Put

(16.14) $$\Phi_j(x_0, \varepsilon, \tau, u^0)=\varphi_j(a^{(3)}, \varepsilon, \tau, U(a^{(3)}, \varepsilon, \tau, x_0, u^0)) + \int_{a^{(3)}}^{x_0} \varepsilon^{-\sigma_j} g_j(x, \varepsilon, \tau, U(x, \varepsilon, \tau, x_0, u^0), \varphi(x, \varepsilon, \tau, u)e^{A(x,\varepsilon)})e^{-A_j(x,\varepsilon)}dx$$

where $(x_0, \varepsilon, \tau, u_1^0, \cdots, u_\alpha^0)$ is an arbitrary point in the domain (16.12) and the integration is to be carried out along the segment Γ_{1x_0} defined in Lemma 5. Then, the mapping $\mathfrak{A}$ is defined as follows:

$$\{\varphi_1, \cdots, \varphi_n\} \to \{\Phi_1, \cdots, \Phi_n\}.$$

Since $\{0, \cdots, 0\} \in F$, F is not empty. Further, it is easy to see that F is convex, closed and normal. On the other hand, since, according to Lemma 5, the value of the function $\max|U_l|^{1/\mu_l}$ decreases montonically as x tends to $a^{(3)}$ on Γ_{1x_0}, the integrand

$$\varepsilon^{-\sigma_j} g_j(x, \varepsilon, \tau, U, \varphi e^{A(x,\varepsilon)})e^{-A_j(x,\varepsilon)}$$

is bounded. Thus, the integral in the right hand side of (16.14) converges and the mapping $\mathfrak{A}$ has a well-defined meaning.

(3°). Existence of a fixed point of $\mathfrak{A}$. Our proof of the main theorem is based on the existence of a fixed-point of the mapping $\mathfrak{A}$.

Proposition 1. *The mapping $\mathfrak{A}$ transforms F into itself.*

Proof. Since the integral in (16.14) are uniformly convergent, the expression $\Phi_j(x, \varepsilon, \tau, u)$ are functions holomorphic and bounded in $(x, \varepsilon, \tau, u)$ (16.12). Therefore, it is sufficient to show

(a) $$|\Phi_j(x_0, \varepsilon, \tau, u^0)| \leqq K_N e^{-Re A_j(x,\varepsilon)} \max_{l=1}^{\alpha} |u_l^0|^{N/\mu_l}.$$

In virtue of the inequalities (16.7) and (16.13), it suffices to prove that

$$|\varphi_j(a^{(3)}, \varepsilon, \tau, x_0, u^0)| + \int_0^{s_0} |\varepsilon|^{\sigma_j^* - \sigma_j}(A_j K_N + B_N)e^{-Re A_j(x,\varepsilon)} \max_{l=1}^{\alpha} |U_l|^{N/\mu_l} ds \leqq K_N e^{-Re A_j(x_0,\varepsilon)} \max_{l=1}^{\alpha} |u_l^0|^{N/\mu_l}$$

where s_0 is the length of the segment Γ_{1x_0}. Since the above inequality is trivial when $s=0$, we have only to show that, on the curve Γ_{1x_0},

$$(16.15)\qquad |\varepsilon|^{\sigma_j^*-\sigma_j}(A_jK_N+B_N)e^{-ReA_j(x,\varepsilon)}\max_{l=1}^{\alpha}|U_l|^{N/\mu_l} \leqq K_N\frac{d}{ds}\left\{e^{-ReA_j(x,\varepsilon)}\max_{l=1}^{\alpha}|U_l|^{N/\mu_l}\right\}.$$

By Lemma 6. this will be an immediate consequence of

$$(16.16)\qquad 4(A_jK_N+B_N)\leqq K_N\left\{N\min_{l=1}^{\alpha}\frac{|\lambda_l|}{\mu_l}\cdot\sin\theta^0\right\},\qquad \sigma_j\leqq\sigma_1,$$

or

$$(16.17)\qquad 4(A_jK_N+B_N)\leqq K_N|\lambda_j|\cdot\sin\theta^0,\qquad \sigma_j>\sigma_1.$$

The inequality (16.16) or (16.17) will be realized by taking K_N sufficiently large if we have

$$(16.18)\qquad 8A_j<N\min_{l=1}^{\alpha}\frac{|\lambda_l|}{\mu_l}\cdot\sin\theta^0,\qquad \sigma_j\leqq\sigma_1,$$

or

$$(16.19)\qquad 8A_j<|\lambda_j|\cdot\sin\theta^0,\qquad \sigma_j>\sigma_1,$$

However, the former is satisfied by taking N sufficiently large and the latter is satisfied also, as we have easily seen from Lemma 5.

Thus we have proved the proposition 1.

Since, the integral in (16.14) being uniformly convergent, $\mathfrak{A}$ is a continuous mapping of F with respect to the topology of uniform convergence (in a wider sense), we can conclude, from Proposition 1, the existence of a fixed point of the mapping $\mathfrak{A}$; namely the system $\{\varphi_1, \cdots, \varphi_n\}$ of F such that

$$\{\varphi_1, \cdots, \varphi_n\}=\{\Phi_1, \cdots, \Phi_n\}.$$

Moreover, this system $\{\Phi_1, \cdots, \Phi_n\}$ is a solution of the system (1.64). In fact, $\{\Phi_1, \cdots, \Phi_n\}$ is a solution of the system

$$\varepsilon^{\sigma_j}\frac{d\Phi_j}{dx}=g_j(x, \varepsilon, \tau, U, \varphi e^{A(x,\varepsilon)})e^{-A_j(x,\varepsilon)}$$

for any $\{\varphi_1, \cdots, \varphi_n\}$ of F; cf. [4].

Thus we have established that there exists, in the family F, a solution of (16.11). We shall denote it by $\psi_{jN}(x, \varepsilon, \tau, U(x, \tau, x_0, u^0))$.

Proposition 2. *The solution* ψ_{jN} *of* (16.11) *such that*

$$\psi_{jN}(x, \varepsilon, \tau, U)=O\left(e^{-A_j(x,\varepsilon)} \max_{l=1}^{\alpha} | U_l |^{N/\mu_l}\right)$$

is unique.

Proof. Assume the contrary, and let $w_j (j=1, 2, \cdots, n)$ be the difference between any two solutions satisfying the same conditions. Then, evidently, from (16.8) and (16.14), the integral inequality

$$(16.20) \qquad | w_j(x_0, \varepsilon, \tau, u^0) | \leqq \int_0^{s_0} | \varepsilon |^{-\sigma_j} A_j \max_{l,x,\varepsilon,\tau} | w_l e^{A_l(x,\varepsilon)} | e^{-ReA_j(x,\varepsilon)} ds$$

must hold. If we put

$$\left\{\max_{l,x,\varepsilon,\tau} | w_l(x, \varepsilon, \tau, U) e^{A_l(x,\varepsilon)} |\right\} \times \left\{\max_{l=1}^{\alpha} | U_l |^{-N/\mu_l}\right\} = M > 0$$

it follows from (16.20) that

$$(16.21) \qquad | w_j(x_0, \varepsilon, \tau, u^0) | \leqq \int_0^{s_0} A_j M | \varepsilon |^{-\sigma_j} e^{-ReA_j(x,\varepsilon)} \max_{l=1}^{\alpha} | U_l |^{N/\mu_l} ds \equiv I_{s_0}.$$

As we have already seen in the proof of Proposition 1, by Lemma 6, we have

$$I_{s_0} \leqq \begin{cases} A_j\left[\dfrac{N \sin \theta^0}{4} \min_{l=1}^{\alpha} \dfrac{| \lambda_l |}{\mu_l}\right]^{-1} M \times e^{-ReA_j(x_0,\varepsilon)} \max_{l=1}^{\alpha} | u_l^0 |^{N/\mu_l}, & \sigma_j \leqq \sigma_1 \\ A_j\left[\dfrac{| \lambda_j | \sin \theta^0}{4}\right]^{-1} M \times e^{-ReA_j(x_0,\varepsilon)} \max_{l=1}^{\alpha} | u_l^0 |^{N/\mu_l} & \sigma_j > \sigma_1 \end{cases}$$

Therefore, by the inequality (16.18) or (16.19) we have

$$| w_j(x_0, \varepsilon, \tau, u^0) e^{-ReA_j(x_0,\varepsilon)} | \times \max_{l=1}^{\alpha} | u_l^0 |^{N/\mu_l} \leqq M/2$$

in any case. It follows that $M < M$ which is a contradiction. Hence $w_j(x, \varepsilon, \tau, u) \equiv 0$ and the uniqueness of the solution is established.

Thus the system (7.1) possesses a solution

$$\Psi_{jN}(x, \varepsilon, \tau, U) = W_{jN}(x, \varepsilon, \tau, U) + \psi_{jN}(x, \varepsilon, \tau, U) e^{A_j(x,\varepsilon)} .$$

To complete the proof of Theorem 3, it remains for us to show that the solution $\Psi_{jN}(x, \varepsilon, \tau, U)$ is independent of N. Since

$$[W_{jN'}(x, \varepsilon, \tau, U) + \psi_{jN'}(x, \varepsilon, \tau, U) e^{A_j(x,\varepsilon)} - W_{jN}(x, \varepsilon, \tau, U)] e^{-A_j(x,\varepsilon)} \quad N < N'$$

is a solution of (16.11) satisfying the condition of Proposition 2, it must be equal to $\psi_{jN}(x, \varepsilon, \tau, U)$ from the above established uniqueness. Therefore we have $\Psi_{jN}(x, \varepsilon, \tau, U) = \Psi_{jN'}(x, \varepsilon, \tau, U)$ which shows that $\Psi_{jN}(x, \varepsilon, \tau, U)$ is independent of of N.

Thus the proof of main theorem has been completed. Q.E.D.

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